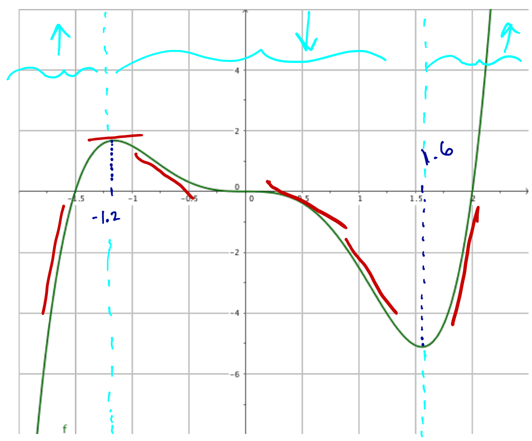
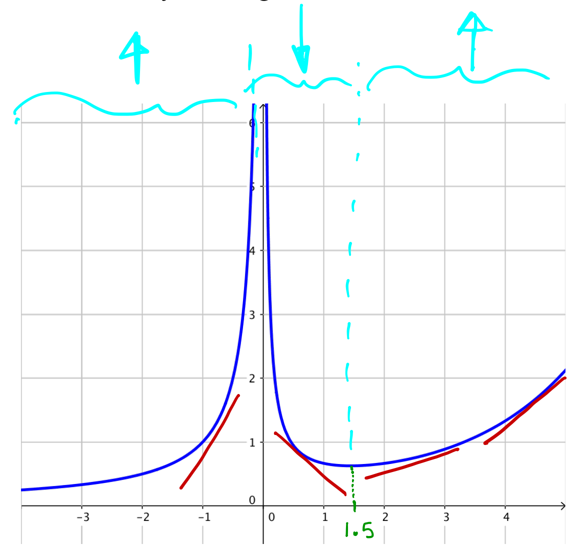
Lecture Notes: 4-3 How Derivatives Affect the Shape of a Graph (part 1)

Motivating Example: For each function graphed below, identify the regions of its *domain* where the function is increasing and where it is decreasing.



$f(x)$ is increasing: $(-\infty, -1.2) \cup (1.6, \infty)$

$f(x)$ is decreasing: $(-1.2, 1.6)$



$f(x)$ is increasing: $(-\infty, 0) \cup (1.5, \infty)$

$f(x)$ is decreasing: $(0, 1.5)$

Question 1: Using language a middle school kid could understand, how would you explain what it means to say a function is *increasing* or *decreasing*?

Read the graph from left to right. ($\longrightarrow$ or as x-values increase)
If graph (or y-values) go up, we say function is increasing.
If graph (or y-values) go down, then decreasing

Question 2: Draw a few sample tangent lines to each graph above. What is the relationship between the slope of the tangent lines and whether the graph is increasing or decreasing?

In red above.
when $f(x)$ is increasing, tangents have positive slope:
when $f(x)$ is decreasing, tangents have negative slope:

Increasing/Decreasing Test +'ve slope

(a) If $f'(x) > 0$ on an interval, then the function $f(x)$ is **increasing** on this interval.

(b) If $f'(x) < 0$ on an interval, then the function $f(x)$ is **decreasing** on this interval.

neg. slope!

These #'s determine the intervals

Choose a "sample" # in each interval

PRACTICE PROBLEM 1: Let $g(x) = 3x^4 - 4x^3 - 12x^2 + 5$.

1. Use the Increasing/Decreasing Test to find the intervals where $g(x)$ is increasing and decreasing.

thinking:
- will find $g'(x)$
- will find when $g'(x) = 0$.
- check on either side of zeros for sign.

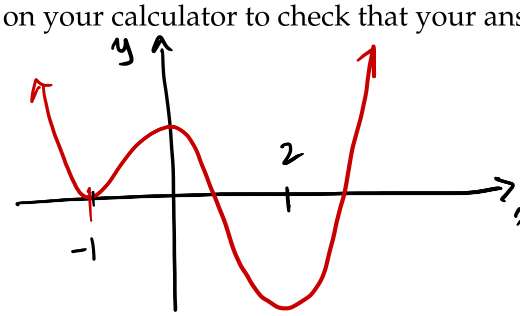
g' has zeros @ $x = 0, 2, -1$.

	-2	-1/2	1	3
interval	$(-\infty, -1)$	$(-1, 0)$	$(0, 2)$	$(2, \infty)$
sign of $g' = 12x(x-2)(x+1)$	$(-)(-)(-) = -$	$(-)(-)(+) = +$	$(+)(-)(+) = -$	$(+)(+)(+) = +$
incr. vs. decr.	decreasing	increasing	decreasing	increasing

ans: $g(x)$ is
- increasing on $(-1, 0) \cup (2, \infty)$
- decreasing on $(-\infty, -1) \cup (0, 2)$

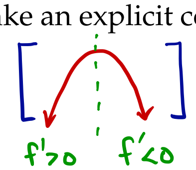
$$g'(x) = 12x^3 - 12x^2 - 24x$$
$$= 12x(x^2 - x - 2)$$
$$= 12x(x-2)(x+1)$$

2. Sketch the graph on your calculator to check that your answer above is correct.



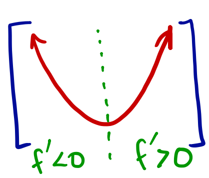
3. What do you observe about the relationship between *local maximums, local minimums* and intervals of *increase* and *decrease*? Make an explicit conjecture.

- $f(x)$ has a local maximum



when f' changes from positive to negative

- $f(x)$ has a local minimum



when f' changes from negative to positive

- We need the function $f(x)$ to be defined at the x-value where $f'(x)$ changes sign, (i.e. the left example is a problem...)

4. Go back and look at the examples at the top of page 1 and see if you need to amend your conjecture above.

QUESTION 3: What is a critical number again?

- an x-value in the domain of $f(x)$ so that (say $x=c$)

(i) $f'(c)=0$ or (ii) $f'(c)$ is undefined.

The First Derivative Test: Suppose that c is a critical number of a continuous function $f(x)$.

a) If $f'(x)$ changes from positive to negative at c, then f has a **local maximum** at c.

b) If $f'(x)$ changes from negative to positive at c, then f has a **local minimum** at c.

c) If $f'(x)$ does not change sign at c, then f has no local maximum or minimum at c.

Using the work from the previous PRACTICE PROBLEM 1, fill in the blanks below for $g(x) = 3x^4 - 4x^3 - 12x^2 + 5$.

(a) $y=0$ is a local minimum of $g(x)$ that occurs at $x=-1$

(b) $y=-27$ is a local minimum of $g(x)$ that occurs at $x=2$

(c) $y=5$ is a local maximum of $g(x)$ that occurs at $x=0$

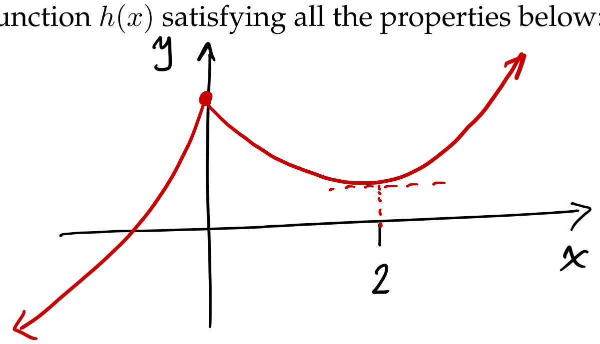
work:

$g(-1) = 3-(-4)-12+5=0$

$g(0)=5$

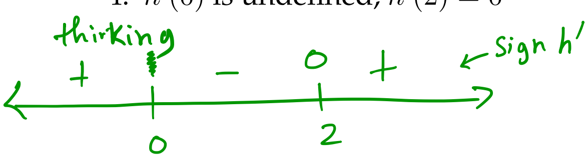
$g(2) = 48-32-48+5$
$= -27$

PRACTICE PROBLEM 2: Sketch the graph of a function $h(x)$ satisfying all the properties below:



1. domain $(-\infty, \infty)$
2. $h'(x) > 0$ on $(-\infty, 0) \cup (2, \infty)$
3. $h'(x) < 0$ on $(0, 2)$
4. $h'(0)$ is undefined, $h'(2) = 0$

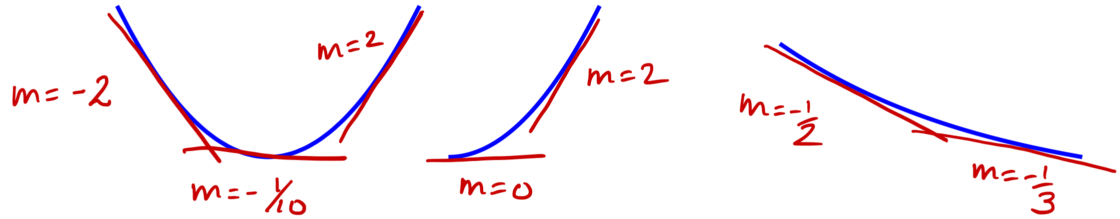
thinking



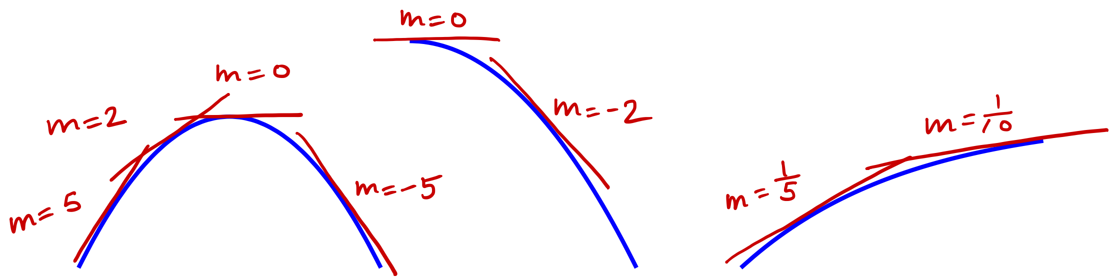
in red

MOTIVATING EXAMPLES: On the sample graphs below, sketch some rough tangent lines. Sketch multiple tangents on each graph. Make rough approximations of the slopes of these tangents.

concave up pictures



concavedown pictures



QUESTION 4: How does the relationship between the tangent line and the graph to which it is tangent differ depending on whether the graph is concave up or concave down?

cc up : tangent below curve

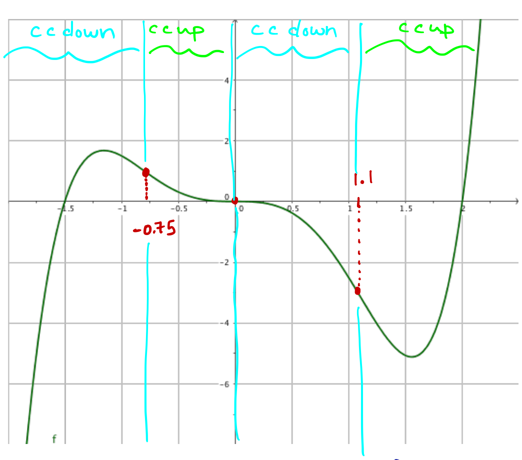
cc down : tangent above curve

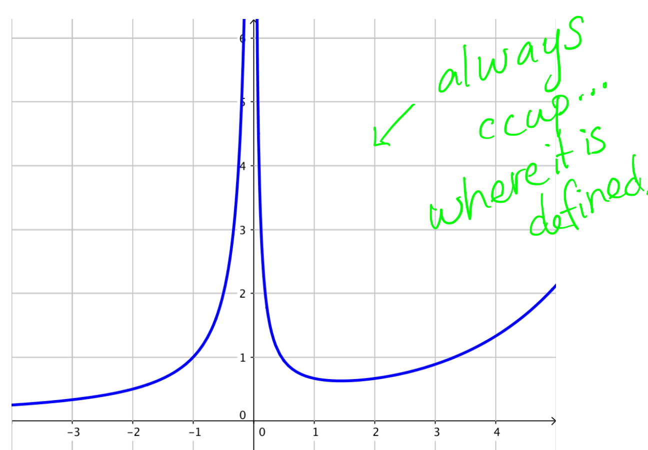
QUESTION 5: If the graphs above are of a function, say $f(x)$, what can you say about its derivative $f'(x)$?

cc up: as x increases, f'(x) increases

cc down: as x increases, f'(x) decreases

QUESTION 6: Estimate the intervals where each function below is concave up and concave down:





$f(x)$ is concave up: $(-0.75, 0) \cup (1.1, \infty)$

$f(x)$ is concave down: $(-\infty, -0.75) \cup (0, 1.1)$

$f(x)$ is concave up: $(-\infty, 0) \cup (0, \infty)$

$f(x)$ is concave down: never

DEFINITION: A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward or vice versa at P.

QUESTION 6: Do the graphs on the previous page have any inflection points?

graph on left: 3 inflection points. marked in red.

graph on right: 0 inflection points.

CONCAVITY TEST & INFLECTION POINTS: Let $f(x)$ be a function defined on an interval I.

a) If $f''(x) > 0$ (that is: f' is increasing) for all x in I, then the graph of f is concave upward on I.

b) If $f'' < 0$ (that is: f' is decreasing) for all x in I, then the graph of f is concave downward on I.

PRACTICE PROBLEM 3: Let $f(x) = 2x^3 - 3x^2 - 12x$. Find the intervals of concavity and the inflection points.

$f'(x) = 6x^2 - 6x - 12$

$f''(x) = 12x - 6 = 0$

$x = \frac{1}{2}$

sample x-values in interval

interval	$(-\infty, \frac{1}{2})$	$(\frac{1}{2}, \infty)$
	0	1
sign of f''	$-$	$+$
concavity	cc up	cc down

when $x = \frac{1}{2}$, $y = f(\frac{1}{2}) = 2(\frac{1}{2})^3 - 3(\frac{1}{2})^2 - 12(\frac{1}{2})$

$= -6.5$

answer:

f is concave up on $(\frac{1}{2}, \infty)$

f is concave down on $(-\infty, \frac{1}{2})$

$(\frac{1}{2}, -6.5)$ is an inflection point.